Hale School

Mathematics Specialist

Term 3 2018

Test 4 - Integration

SECTION ONE

Name: ANT. I. DIFF

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Instructions:

- **SECTION ONE: Calculators are NOT allowed**
- **External notes are not allowed**
- **Duration of SECTION ONE: 25 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

Question 1 **(8 marks)**

Determine the following integrals:

a) $\int \sin^3(x)\cos^3(x)\,dx$ (4 marks)

$= \int \sin x(1-\cos^2 x)\cos^3 x\,dx$ ✓ separates $\sin x$

$= \int \sin x\cos^3 x - \sin x\cos^5 x\,dx$ ✓ rearranges

$= -\frac{\cos^4 x}{4} + \frac{\cos^6 x}{6} + c$ ✓ $\frac{\cos^4 x}{4}$ and $\frac{\cos^6 x}{6}$

✓ signs and $+c$

Or $\int \left(\frac{1}{2}\sin 2x\right)^3 dx = \frac{1}{8}\int \sin 2x(1-\cos^2 2x)\,dx$

$= -\frac{\cos 2x}{16} + \frac{\cos^3 2x}{48} + c$

Or $\int \cos x(\sin^3 x)(1-\sin^2 x)\,dx = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + c$

b) $\int_0^{1/2} \frac{\cos(\pi x)}{2+\sin(\pi x)}\,dx$ (4 marks)

$= \left[\frac{1}{\pi}\ln(2+\sin\pi x)\right]_0^{\frac{1}{2}}$ ✓ recognises $\frac{f'(x)}{f(x)}$

$= \frac{1}{\pi}\ln 3 - \frac{1}{\pi}\ln 2$ ✓ factor of $\frac{1}{\pi}$

✓ substitutes $0, \frac{1}{2}$

$= \frac{1}{\pi}\ln \frac{3}{2}$ ✓ simplifies

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Question 2 **(6 marks)**

Using the substitution $u = \tan x$ and the identity $\sec^2 x = 1 + \tan^2 x$ determine the following definite integral:

$$\int_{\pi/4}^{\pi/3} \tan^2 x + \tan^4 x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$x = \pi/4 \quad u = 1$

$x = \pi/3 \quad u = \sqrt{3}$

$$= \int_1^{\sqrt{3}} \tan^2 x + \tan^4 x \cdot \frac{du}{\sec^2 x}$$

$$= \int_1^{\sqrt{3}} \frac{u^2 + u^4}{1+u^2} \, du$$

$$= \int_1^{\sqrt{3}} \frac{u^2(1+u^2)}{1+u^2} \, du$$

$$= \left[u^3/3 \right]_1^{\sqrt{3}}$$

$$= \frac{(\sqrt{3})^3}{3} - \frac{1}{3}$$

$$= \sqrt{3} - \frac{1}{3}$$

✓ $du = \sec^2 x \, dx$

✓ $u = 1, \sqrt{3}$

✓ correct integral

✓ simplifies to u^2

✓ $u^3/3$

✓ $\sqrt{3} - \frac{1}{3}$

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Question 3 **(6 marks)**

a) Express $\frac{2x^2 - 9x + 12}{(x-2)(x-3)}$ in the form $A + \frac{B}{x-2} + \frac{C}{x-3}$ (3 marks)

$$\frac{2x^2 - 9x + 12}{(x-2)(x-3)} = A + \frac{B}{x-2} + \frac{C}{x-3}$$

$$\Rightarrow 2x^2 - 9x + 12 = A(x-2)(x-3) + B(x-3) + C(x-2)$$

✓ rearranges

Compare x^2 terms: $A = 2$

Consider $x = 2$: $2 = -B \quad \therefore B = -2$

Consider $x = 3$: $3 = C \quad \therefore C = 3$

✓ $A = 2$

✓ $B = -2$, $C = 3$

$$\frac{2x^2 - 9x + 12}{(x-2)(x-3)} = 2 - \frac{2}{x-2} + \frac{3}{x-3}$$

b) Hence determine $\int_4^5 \frac{2x^2 - 9x + 12}{(x-2)(x-3)} dx$ (3 marks)

$$= \int_4^5 2 - \frac{2}{x-2} + \frac{3}{x-3} \, dx$$

$$= \left[2x - 2\ln(x-2) + 3\ln(x-3) \right]_4^5$$

$$= (10 - 2\ln 3 + 3\ln 2) - (8 - 2\ln 2 + 3\ln 1)$$

$$= 2 - 2\ln 3 + 5\ln 2$$

$$\left(= 2 + \ln \frac{32}{9} \right)$$

✓ integrates

✓ substitutes 4, 5

✓ simplifies

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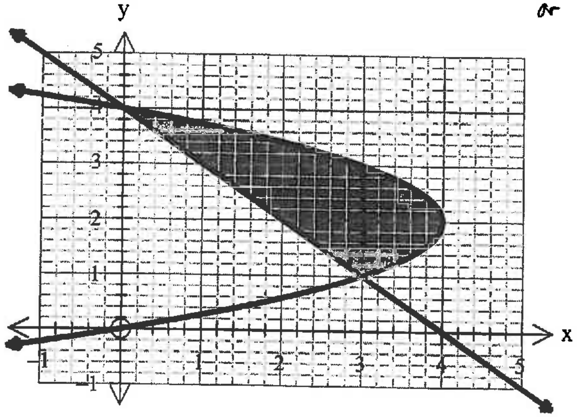
Question 4 **(6 marks)**

The graphs defined by $(y-2)^2 = 4-x$ *and* $x+y=4$ are shown below. Calculate the **exact** area enclosed between the two curves as shaded in the diagram below.



or $\int_0^3 2+\sqrt{4-x}-(4-x)\,dx$

$+ \int_3^4 2+\sqrt{4-x}-(2-\sqrt{4-x})\,dx$

$= \left[-2x+\frac{x^2}{2}-\frac{2}{3}(4-x)^{3/2}\right]_0^3$

$+ \left[-\frac{4}{3}(4-x)^{3/2}\right]_3^4$

$= \left(-6+4.5-\frac{2}{3}\right)+\frac{16}{3}$

$+ \left(0+4/3\right)$

$= -6+4.5+18/3 = \underline{4.5 \text{ units}^2}$

$\text{Area} = \int_1^4 x_1 - x_2 \, dy$ ✓ method

$= \int_1^4 4-(y-2)^2 - (4-y)\,dy$ ✓ integrand

$= \int_1^4 y - (y-2)^2\,dy$ ✓ limits

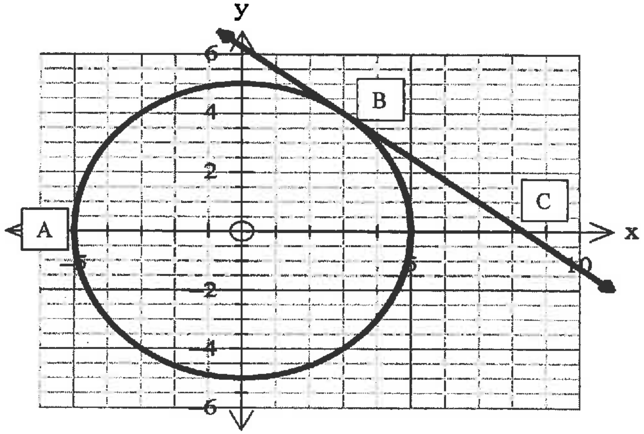
$= \left[y^2/2 - \frac{(y-2)^3}{3}\right]_1^4$ ✓ integrates

$= \left(8 - 8/3\right) - \left(\frac{1}{2}+\frac{1}{3}\right)$ ✓ substitutes

$= 8 - 3 - \frac{1}{2}$

$= \underline{4\frac{1}{2}}$ units^2 ✓ evaluates

Question 5 **(5 marks)**



The graph above shows the circle $x^2+y^2=25$ and the line $3x+4y=25$ which is a tangent to the circle, touching at point B. Points A and C are x – intercepts for the circle and the line respectively.

The region bounded by the minor arc AB, line segment BC and the x – axis is rotated 360° about the x – axis.

Determine the volume of the resulting solid accurate to 0.1 cubic units.

Intersection of $\left.\begin{array}{l} x^2+y^2=25 \\ 3x+4y=25 \end{array}\right\}$ $x=3,\ y=4$ ✓ B

B is $(3,4)$

$\text{Volume} = \int_{-5}^{3} \pi(25-x^2)\,dx + \int_3^{25/3} \pi\cdot\left(\frac{25-3x}{4}\right)^2 dx$ ✓ limits

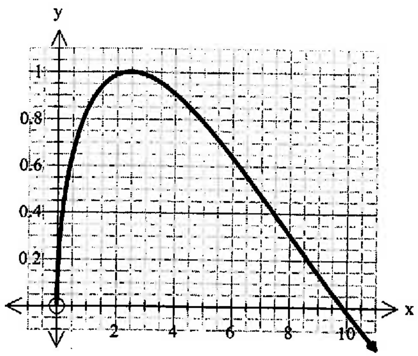
✓ 1st integral

$= \frac{1600\pi}{9}$ ✓ 2nd integral

$= 558.5$ units^3 (1dp)

✓ evaluates

Question 6 **(7 marks)**



The diagram opposite shows the graph of the function $y = \sin(\sqrt{x})$.

A is the area of the region between the curve and the x – axis.

a) Write down an integral for the value of A and calculate this value to 5 decimal places. (2 marks)

$A = \int_0^{\pi^2} \sin(\sqrt{x})\,dx = 6.28319$ (5dp) ✓ integral and limits

6.28184 ✓ evaluation

b) Estimate the value of A using 6 midpoint rectangles and state the percentage error for this result accurate to 0.1%. (2 marks)

Estimated value = 6.41976 (5dp) ✓ estimated value

6.42098

% error = 2.2% ✓ % error

c) Investigate the number of strips required using midpoint rectangles so that the percentage error between the estimated value and the true result is less than 1%. Show evidence for your answer. (3 marks)

With 10 strips – % error is 1.01%

1.03%

✓ value for 10

With 11 strips – % error is 0.88%

0.89%

✓ value for 11

∴ need 11 strips ✓ conclusion

END OF TEST